Probability

\[ P(\text{win}) = P(\text{first ball drawn in one of your chosen numbers}) \times P(\text{2nd ball drawn in one of your remaining chosen numbers | 1st ball drawn being one of your chosen numbers}) \times P(\text{3rd ball drawn in one of your remaining chosen numbers | 1st and 2nd balls were two of your chosen numbers}) \times \ldots \times P(\text{6th ball drawn in your last chosen number | 1st, 2nd, 3rd, 4th, and 5th balls were 4 of your chosen numbers}) \times P(\text{your powerball choice is i | the number drawn}) \]

\[ = \frac{5}{69} \times \frac{4}{68} \times \frac{3}{67} \times \frac{2}{66} \times \frac{1}{65} \times \frac{1}{26} \]

\[ = \frac{1}{292,201,338} \]

\[ \approx 0. \]
Note: the 5 balls are dependent events (each drawing depends on all the earlier draws).

The powerball is independent

Example:

Draw 2 cards off the top of a well-shuffled deck.

Prob. that the 2 cards are both aces.

4 out of 52 aces.

$$p(1^{\text{st}} \text{ card is A}) = \frac{4}{52}.$$

Given that 1st card is A, 3 aces left out of 51 cards.

$$p(2^{\text{nd}} \text{ card is A} \mid 1^{\text{st}} \text{ card was A}) = \frac{3}{51}.$$

$$P(1^{\text{st}} \text{ 2 cards are both A}) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652}.$$
Toss coin twice.

What's the chance of HT?

\[ P(H \text{ on 1st toss}) = \frac{1}{2}, \]
\[ P(T \text{ on 2nd toss}) = \frac{1}{2}. \]

\[ = P(T \text{ on 2nd toss } | H \text{ on 1st toss}) \]

For the coins, what happened on earlier events does not affect later events.

They are independent.

Two events are independent if the chances for the 2nd event given the first are the same, regardless of the outcome of the 1st event.

Otherwise, they are dependent.
two draws, at random, with replacement

1st draw was a 1, what's the chance of 12 on 2nd draw?

\[ P(12 \text{ on 2nd draw}) = \frac{2}{5} \]

\( \Rightarrow \text{independent events} \)

when drawing \underline{with replacement}, the content of the box is the same for every draw.

If \( n \) draws are made \underline{without replacement}

After 11 is drawn on 1st draw, box now contains

\[ | 1 | 2 | 2 | 3 | \]

\[ P(12 \text{ on 2nd draw} \mid 11 \text{ on 1st draw}) = \frac{2}{4} = \frac{1}{2} \]

\( \Rightarrow \text{dependent events} \)
Some Notation.

- Event $A$
- Probability of event $A$ is written $P(A)$
- Events $A$ and $B$
- Conditional probability of event $B$ given event $A$ = $P(B|A)$

Multiplication Rule.

\[
P(A \text{ and } B) = P(B|A)P(A) = P(A|B)P(B)
\]

Independence.

Requires that
\[
P(B|A) = P(B)\]
\[
P(A|B) = P(A)
\]

Implies that
\[
P(A \text{ and } B) = P(A) \times P(B)
\]
for independent events
Probability of an event = \frac{\# \text{ outcomes corresponding to that event}}{\text{total \# outcomes}}

when each outcome is equally likely.

ie list the possible outcomes.

count the relevant ones.

g Roll two dice.

6 possible outcomes on 1st die

for each of these, there are 6 possible outcomes on 2nd die.

\Rightarrow 36 \text{ possible outcomes in total.}
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<th></th>
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<td>6,6</td>
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</table>

\[ P(\text{sum of the two rolls will be 2}) \]
- only one possibility \((1,1)\)
  \[ \Rightarrow \text{prob. is } \frac{1}{36} \]

\[ P(\text{sum will be 7}) \]
- 6 possibilities \((6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\).
  \[ \Rightarrow \text{prob. is } \frac{6}{36} = \frac{1}{6} \]
\[ P( \text{1st die has more pips than 2nd die}) \]

- This corresponds to all the entries in the bottom-left part of the table.

- There are 15 ways that the 1st die shows more spots than 2nd die.

\[ \text{Prob } = \frac{15}{36} \]

\[ P( \text{Red die showing more pips} \mid \text{White die showing 4 pips}) \]

\[ \text{Restricted to those outcomes to which fit the condition.} \]

- There are now 6 outcomes that we're interested in, and two of them (5,4) (6,4) have more spots on red than on white.

\[ \Rightarrow \text{Prob } = \frac{2}{6} = \frac{1}{3} \]
Recall multiplication rule for events like:

\[ p(\text{1st ball is blue and 2nd ball is green}) \]

Now, look at "or" case.

Event A happens or B happens or both

Start by looking at the case where it is impossible for both events to happen together.

Such events are called mutually exclusive

Two events are mutually exclusive when the occurrence of one prevents the occurrence of the other.

Eg. Roll 2 dice.

Event A: Sum is less than 3
Event B: Sum is greater than 6

Events A and B cannot happen together

-They are mutually exclusive.
A card is dealt

A: card is Heart.
B: card is Spade

- it clearly can't be both at the same time.

If two events are **mutually exclusive**

the chance that at least one occurs

is the sum of the chance of

each.

**Example.**

What's the chance of the top card
being either a H or a S?

\[
\text{chance } \frac{13}{52} = \frac{1}{4} \quad \text{that it is H,}
\]

\[
\frac{13}{52} = \frac{1}{4} \quad \text{S}
\]

It **cannot** be both \( \Rightarrow \) mutually exclusive events

\[
P(H \cup S) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]
Example

Roll 2 dice

what's the chance of at least one \(\square\)?

chance of \(\square\) on red die is \(\frac{1}{6}\)

chance of \(\square\) on white die is \(\frac{1}{6}\)

    total \(\frac{1}{3}\)

From the table of outcomes when we roll 2 dice

chance of at least one \(\square\) is \(\frac{11}{36}\).

The two events (\(\square\) on red, \(\square\) on white) are not mutually exclusive.

they can both occur at once \((1,1)\)

addition rule overcounts due to this.

- double count the \((1,1)\) outcome.

If two events are not mutually exclusive

the addition rule gives a result that is too big.
To correct for double counting, subtract the probability that both happen.

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

\[ P(\text{1 or a red die} \text{ or } \text{6 or a white die}) \]

\[ = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} \]

\[ = \frac{11}{36} \]

mutually exclusive vs independent

\[ \text{If } A \text{ happens, } B \text{ cannot.} \]

\[ \text{Knowing that } A \text{ has happened does not change the probability that } B \text{ will happen.} \]

mutually exclusive events are not independent.

\[ \text{Knowing that one has happened tells you a lot (everything) about the probability of the other event.} \]
\[ p(A), \]
\[ p(\neg A) \text{ "not } A" \]
\[ p(A) + p(\neg A) = 1. \text{ one of } A \text{ or its complement must happen.} \]

Roll 4 dice.

What's the chance that at least one \[ \Box \Box \] shows?

\[ \text{If not } \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \]
\[ \text{& the events } 4 \text{ are not mutually exclusive.} \]

Can list all possible outcomes:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 3 \\
1 & 1 & 1 & 4 \\
1 & 1 & 1 & 5 \\
1 & 1 & 1 & 6 \\
1 & 1 & 2 & 1 \\
1 & 1 & 2 & ? \\
1 & 1 & 2 & ? \text{ etc} \\
\end{array}
\]

but its tedious + error prone

\[ \text{etc} \]
Event A is "at least one 1" shows. What is the complementary event?

- no 1's show,

not a on 1st die is independent of not a on 2nd die.

\[ P(\text{not a 1}) = \frac{5}{6} \]

\[ P(\text{1st die is not 1 AND 2nd die is not 1}) = \frac{5}{6} \times \frac{5}{6} \]

The events are independent.

\[ P(\text{no 1's on 4 dice}) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \]

The event we're interested in is the complementary event to "no 1 on 4 dice".

\[ \Rightarrow \text{prob. of at least one 1 or 4 dice} \]

is \[ 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \]

\[ = 1 - 0.482 \]

\[ = 0.518 \]
If we're looking for chance of "at least one something"
- compute chance of "no somethings"
- desired prob is $1 - \text{ chance of "no somethings"}$

\[
\begin{align*}
P_1 \rightarrow P_2 \rightarrow P_3 \\
\text{alarm clock goes off} \quad \text{short line at coffee shop} \quad \text{no flat tire on bicycle} \quad \text{arrive on time to class}
\end{align*}
\]

If all components are **independent**, the **system works when all the components work**.

all the components work with prob. $P_1 \times P_2 \times P_3$
System works if any one of the components works.

Easiest to consider the case of the system not working
- when all the components do not work.
- the components are independent.

\[ \Rightarrow \text{system doesn't work with prob} \quad (1-P_1) \times (1-P_2) \times (1-P_3) \]

\[ \Rightarrow \text{system works with prob} \quad 1 - (1-P_1)(1-P_2)(1-P_3) \]
Summary.

Event A:

\[ 0 \leq P(A) \leq 1 \]

Opposite event

\[ P(\text{not } A) = 1 - P(A) \]

Addition

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

(for both)

If \( A, B \) are mutually exclusive

Multiplication

\[ P(A \text{ and } B) = P(A | B) P(B) \]
\[ = P(B | A) P(A) \]

for independent events

\[ = P(A) P(B) \]